Mu Sigma

The Battle of Trafalgar – Mathematical modelling

Thursday Learning Hour – 8th July 2021

Do The Math

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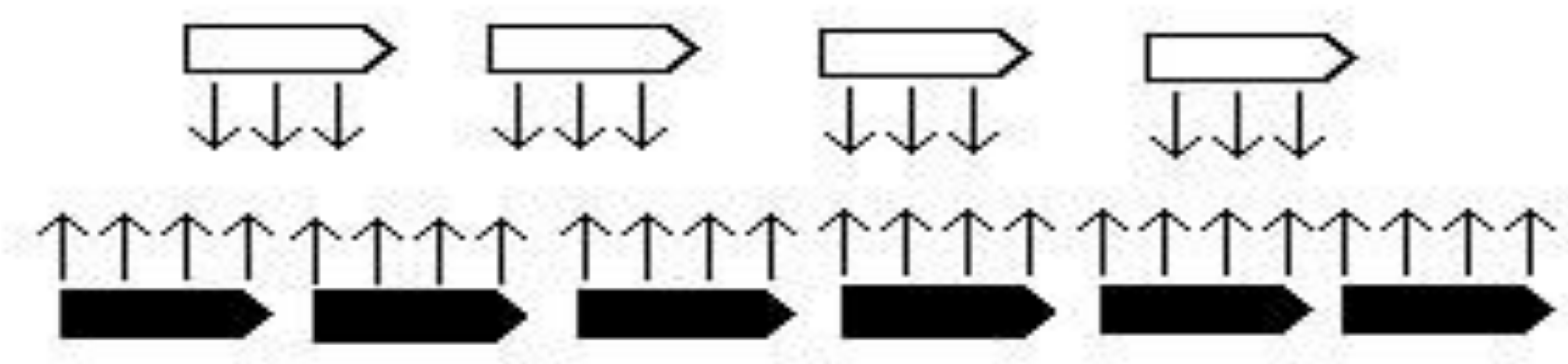
Background information

Battle of Trafalgar, (October 21, 1805), naval engagement of the Napoleonic Wars, which established British naval supremacy for more than 100 years; it was fought west of Cape Trafalgar, Spain, between Cádiz and the Strait of Gibraltar. A fleet of 33 ships (18 French and 15 Spanish) under Admiral Pierre de Villeneuve fought a British fleet of 27 ships under Admiral Horatio Nelson.

Naval ships during 1800s and their huddles

- Firing on the enemy ship is done from the side
- Due to poor visibility, the enemy ships were spotted when they were 10 miles away.
- Communication is limited as it is done by raising flags. The ships immediately behind can see and need to communicate to the next by raise of flags (code signals).
- Speed of the sail depends on wind speed.
- Limited information on enemies' strategies and the number of ships needs to be estimated.
- During Battle, the visibility is poor due to smoke.

Conventional Naval Battle



Suppose French and British naval capability was comparable and assume that attrition rate is 10%

At any time ‘n’

$$F_{n+1} = F_n - 0.1 \times B_N$$

$$B_{n+1} = B_n - 0.1 \times F_N$$

initial condition

F_0= 33

B_0= 27

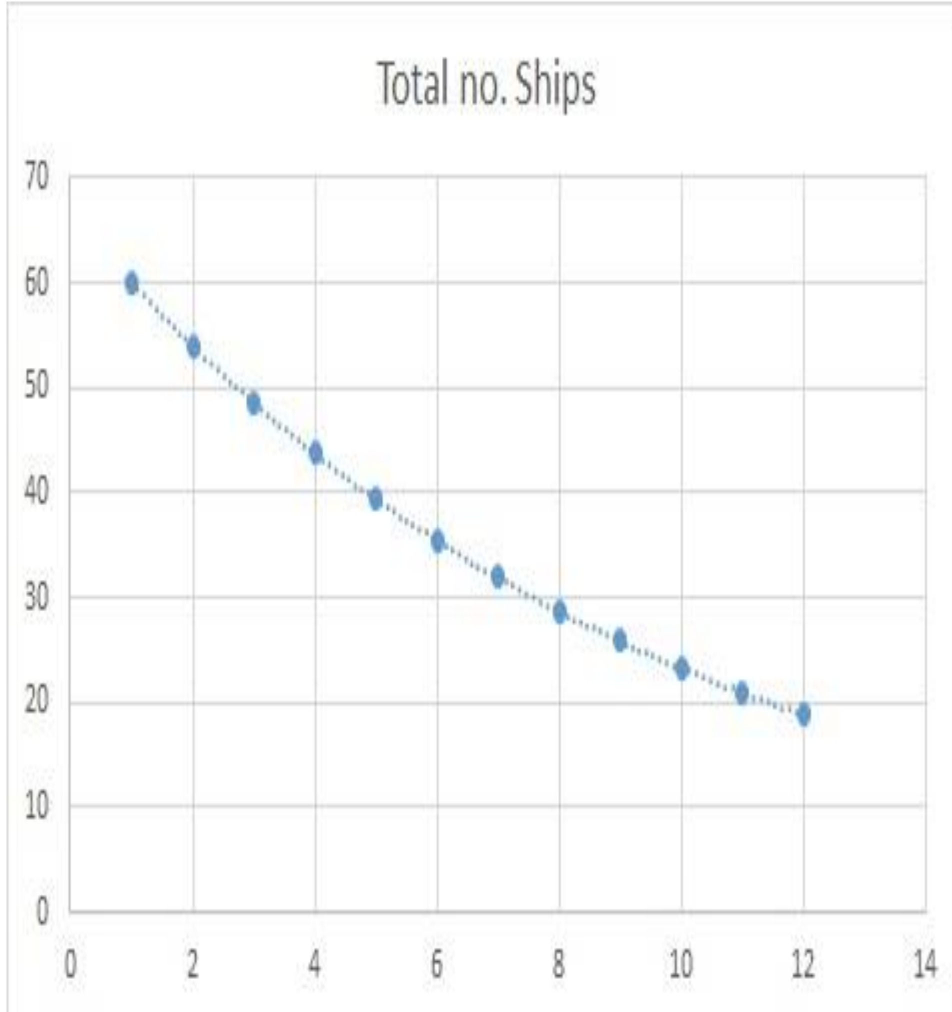
We find that at the end of the battle B gets defeated and F is left with 18 ships.

(and not 33-27 = 6 ships)

Attrition rate k =		0.1
Time stages	F	B
Initial	33	27
1	30.3	23.7
2	27.93	20.67
3	25.863	17.877
4	24.0753	15.2907
5	22.54623	12.88317
6	21.25791	10.62855
7	20.19506	8.502756
8	19.34478	6.48325
9	18.69646	4.548772
10	18.24158	2.679126
11	17.97367	0.854968
12	17.88817	-0.9424

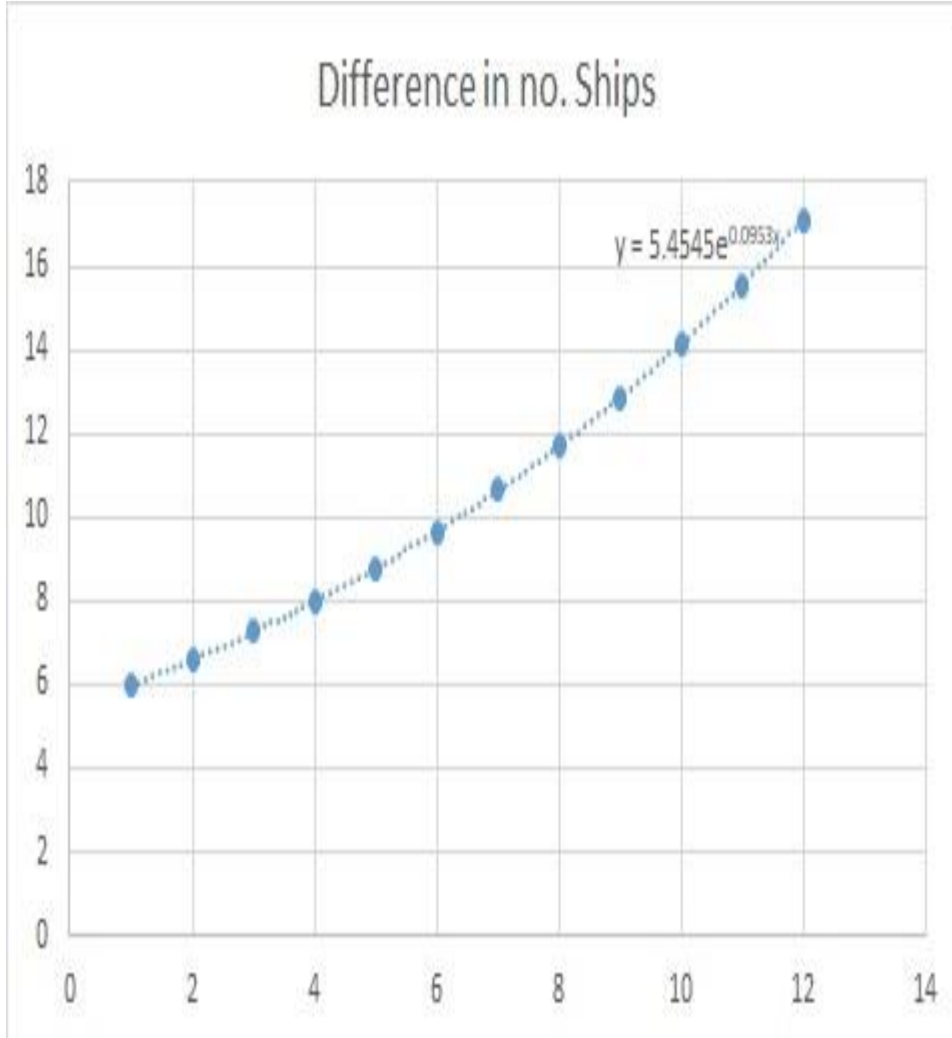
The total number of ships surviving at any time follows exponential decay

Time stages	F	B	Total no. Ships
Initial	33	27	60
1	30.3	23.7	54
2	27.93	20.67	48.6
3	25.863	17.877	43.74
4	24.0753	15.2907	39.366
5	22.54623	12.88317	35.4294
6	21.25791	10.62855	31.88646
7	20.19506	8.502756	28.697814
8	19.34478	6.48325	25.8280326
9	18.69646	4.548772	23.24522934
10	18.24158	2.679126	20.92070641
11	17.97367	0.854968	18.82863577



Difference in the number of ships at anytime is an exponential growth

Time stages	F	B	Difference in no. Ships
Initial	33	27	6
1	30.3	23.7	6.6
2	27.93	20.67	7.26
3	25.863	17.877	7.986
4	24.0753	15.2907	8.7846
5	22.54623	12.88317	9.66306
6	21.25791	10.62855	10.629366
7	20.19506	8.502756	11.6923026
8	19.34478	6.48325	12.86153286
9	18.69646	4.548772	14.14768615
10	18.24158	2.679126	15.56245476
11	17.97367	0.854968	17.11870024



Let us solve it mathematically

$F = number\ of\ ships\ France\ have\ at\ any\ given\ time\ t$

$B = number\ of\ ships\ France\ have\ at\ any\ given\ time\ t$

The rate at which each side loses ships is proportional to the number of ship the opponent have.

$$\frac{dF}{dt} = -bB,\ b > 0 \quad \text{---}(1)$$

$$\frac{dB}{dt} = -aF\ ,\ a>0 \quad \text{---}(2)$$

a and b are attrition rates for France and Britain respectively. It is same as the proportionality constants. a and b are positive.

The negative sign denotes loss or attrition

Solving for a special case

Assuming that the capabilities of F and B are equal or comparable.

Case (i), $a = b = k$ *(say)*

$$\frac{dF}{dt} = -kB, k > 0 \qquad \text{---}(3)$$

$$\frac{dB}{dt} = -kF \;, a>0 \qquad \text{---}(4)$$

Adding (3) and (4)

$$\frac{d(F+B)}{dt} = -k(F+B), k > 0$$

Solving by variable separable method

$$\frac{d(F+B)}{(F+B)} = -kdt, k > 0$$

$$\int \frac{d(F+B)}{(F+B)} = -k \int dt$$

$$ln(F + B) = -kt + c$$

$$(F + B) = e^{-kt + c}$$

$$(F + B) = e^{C} * e^{-kt} \text{ --------------------------------(5)}$$

When t= 0, Let the initial values of F and B be F_0 and B_0 , equation (5) becomes

$$F_0 + B_0 = e^{C}. \text{ --(6)}$$

Now equation (5) becomes,

$$\boldsymbol{(F(t) + B(t)) = (F_0 + B_0)e^{-kt}} \text{ ------------------(7)}$$

insight: *The total number of ships over time is an exponential decay. Here $F_0 = 33$ and $B_0 = 27$*

Let us compute the difference in the number of ships

$$\frac{dF}{dt} = -bB, b > 0 \quad \text{---(1)}$$

$$\frac{dB}{dt} = -aF \;\; , a>0 \quad \text{---(2)}$$

Case (i), a = b = k (say)

$$\frac{dF}{dt} = -kB, k > 0 \quad \text{---(3)}$$

$$\frac{dB}{dt} = -kF \;\; , a>0 \quad \text{---(4)}$$

Subtracting (3) from (4)

$$\frac{d(F-B)}{dt} = k(F-B), k > 0$$

Solving by variable separable method

$$\frac{d(F-B)}{(F-B)} = kdt, k > 0$$

$$\int \frac{d(F-B)}{(F-B)} = k \int dt$$

$$ln(F - B) = kt + c_1$$

$$(F - B) = e^{kt + c_1}$$

$$(F - B) = e^{c_1} * e^{kt} \text{ --------------------------------(8)}$$

Continued…

When $t = 0$,

Let the initial values of F and B be F_0 *and* B_0 *, equation (5) becomes*

$F_0 - B_0 = e^{c_1}$. --(9)

Now equation (5) becomes,

$\left(\boldsymbol{F}(\boldsymbol{t}) - \boldsymbol{B}(\boldsymbol{t})\right) = (\boldsymbol{F_0} - \boldsymbol{B_0})\boldsymbol{e}^{\boldsymbol{kt}}$ ------------------(10)

insight: *The difference in the number of ships over time is an exponential growth.*

Here $F_0 = 33$ *and* $B_0 = 27$

Let us solve for F and B

$$(\boldsymbol{F}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t}))=(\boldsymbol{F_0}+ \boldsymbol{B_0})\boldsymbol{e}^{-\boldsymbol{kt}} \quad \text{------------------------}(7)$$

$$(\boldsymbol{F}(\boldsymbol{t}) - \boldsymbol{B}(\boldsymbol{t}))=(\boldsymbol{F_0} - \boldsymbol{B_0})\boldsymbol{e}^{\boldsymbol{kt}} \quad \text{---------------------------}(10)$$

Solving for F(t) and B(t)

$$F(t) = \frac{1}{2}\left((F_0+ B_0)e^{-kt} +(F_0 - B_0)e^{kt}\right) \text{-----------}(11)$$

$$F(t) = (F_0 Cosh(kt) - B_0 Sinh(kt)\,) \text{------------------}(11A)$$

$$B(t) = \frac{1}{2}\left((F_0+ B_0)e^{-kt} -(F_0 - B_0)e^{kt}\right) \text{------------}(12)$$

$$B(t) = (B_0 cosh(kt) - F_0 Sinh(kt) +) \text{---------------}(12A)$$

Here $F_0 = 33$ *and* $B_0 = 27$

No. of ships on either sides follow hyperbolic functions

Time stages	F	B
Initial	33	27
1	30.3	23.7
2	27.93	20.67
3	25.863	17.877
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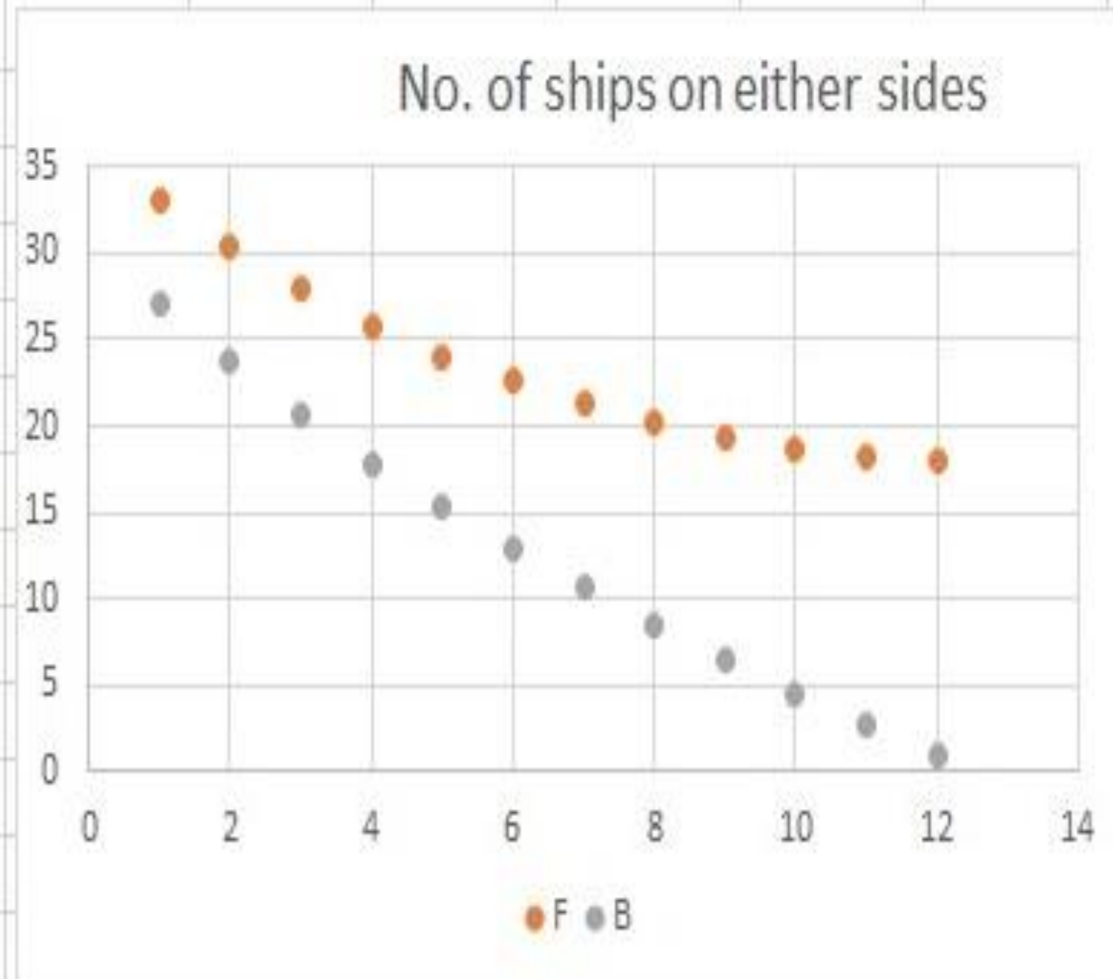


What if the forces were not equal in strength?

$\frac{dF}{dt} = -bB$, $b > 0$ ---(1)

$\frac{dB}{dt} = -aF$, $a>0$ ---(2)

Case (ii), $a \neq b$,

From (1) on differentiation, we get,

$\frac{d^2F}{d^2t} = -b\frac{dB}{dt}$ ---(13)

$\frac{d^2F}{d^2t} = abF$ ---(14)

Auxiliary equation is $m^2 = ab$

$m = \pm\sqrt{ab}$

Solving the ODE

$F(t) = \alpha_1 e^{\sqrt{ab}t} + \alpha_2 e^{-\sqrt{ab}t}$ ------------------------------------*(15)*

Substituting this in

$\frac{dB}{dt} = -aF$, we get

$$\frac{dB}{dt} = -a\left(\alpha_1 e^{\sqrt{ab}t} + \alpha_2 e^{-\sqrt{ab}t}\right)$$

On integrating w.r.t. we get

$B(t) = -\left(\alpha_1 \sqrt{\frac{a}{b}} e^{\sqrt{ab}t} + \alpha_2 \sqrt{\frac{a}{b}} e^{-\sqrt{ab}t}\right)$ --------------------------*(16)*

Where the constants can be found using initial conditions

Finding the constants

$$F_0 = \alpha_1 + \alpha_2$$

$$B_0 = -\left(\alpha_1\sqrt{\frac{a}{b}} + \alpha_2\sqrt{\frac{a}{b}}\right)$$

Alternate ways of solving

$$\frac{dF}{dt} = -bB,\ b > 0$$

$$\frac{dB}{dt} = -aF,\ a>0$$

$$\frac{dF}{dB} = \frac{\frac{dF}{dt}}{\frac{dB}{dt}} = \frac{bB}{aF}$$

$$\frac{dF}{dB} = \frac{bB}{aF}$$

$$aFdF = bBdB$$

$$a\int FdF = b\int B\,dB$$

$$aF^2 = bB^2$$

Insight: it's following a square law. Remote chance of British winning the battle by conventional naval battle.

Nelson's strategy – Divide and conquer

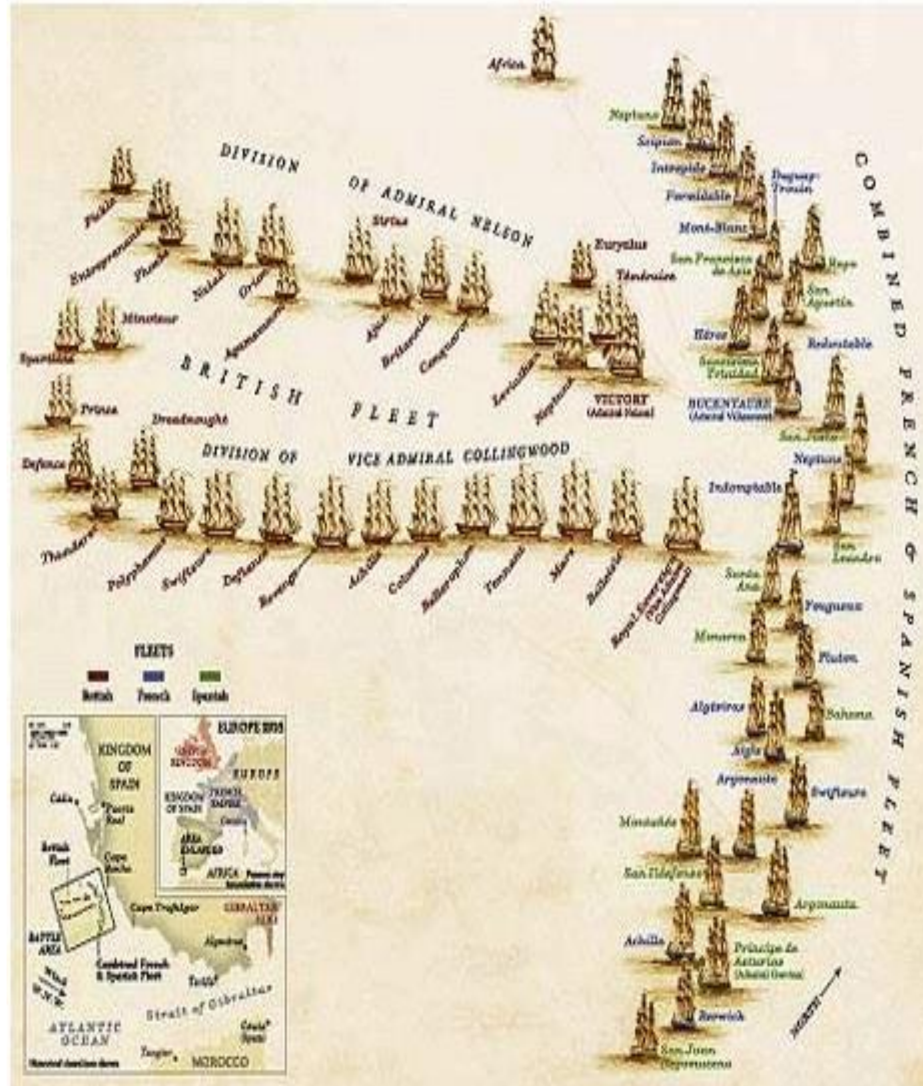
concentrate force at a particular point

Head on strike

Split the enemy fleet

Part of the French ship could not engage in the battle for a long time as they need to turn the ship.

The square law applies.



About Lord Admiral Nelson.

A great Motivator. "England expects that every man will do his duty“- said during the battle

A charismatic Leader

A strategist – he planned the battle and communicated the plan to all. So there was no need to communicate during the battle. The men executed the plan even without his command (he was wounded during the battle). He took the non-conventional way of attacking, French new it but were afraid to implement it.

At 4:30 pm before his death, after the battle was won, he commanded all the ships to be anchored, he had predicted a storm and had planned to win the battle before the storm and save the ships.

A good Forecaster – He forecasted the number of ships the enemies would have.

A loyalist- He created a sense of loyalty among his men. His last words were “Thank God I have done my duty.

Neither Nelson nor his team new calculus, but he had the optimal solution.

What happened actually?

- In five hours of fighting, the British devastated the enemy fleet, destroying 19 enemy ships.
- **No British ships** were lost, but 1,500 British seamen were killed or wounded in the heavy fighting.
- The battle raged at its fiercest around the *Victory,* and a French sniper shot Nelson in the shoulder and chest. The admiral was taken below and died about 30 minutes before the end of the battle. Nelson's last words, after being informed that victory was imminent, were "Now I am satisfied. Thank God I have done my duty."
- https://www.britishbattles.com/napoleonic-wars/battle-of-trafalgar/

Insight: No British ship were lost – this means our math is wrong or not ok. Should we consider the human factors, state of nature and other agents?

Lessons learnt from Battle of Trafalgar

- https://www.mediate.com/articles/conbere.cfm
- https://www.youtube.com/watch?v=TAfSpx8ynLE
- https://towardsdatascience.com/run-your-data-science-team-like-an-admiral-f2f313a837fe

Open question

There were so many other factors that influenced the outcome of the battle, like wind, superior Naval power, Leadership traits of the commanders, timing and many more

Question: Was the battle won because of strategy or leadership traits of Nelson.

That is any commander, with this strategy would have won the battle. Nelson reduced the actualities

Is the above hypothesis, correct?

Try solving the problem by Agent Based Model.

https://github.com/doolanshire/trafalgar

https://ieeexplore.ieee.org/document/1231440

https://eoc.mu-sigma.com/search/tree/Foundation/dcaf53b2-a363-44e8-9952-038b835c03f6

Thank You